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HARMONIOUS COLORING OF SOME CLASSES OF DIRECTED TREES

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Abstract. The proper harmonious coloring number of a digraph $\overrightarrow{D}$ is the least number of colors needed to label the vertices of $\overrightarrow{D}$ such that adjacent vertices receive different colors and no two arcs are incident with the same pair of colors. In this paper we investigate the proper harmonious coloring number of certain classes of directed trees such as bistar, centipede and banana tree.

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1. INTRODUCTION

For all notations in graph theory we follow Harary [1] and Chartrand [2]. Vertex coloring is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color and an edge coloring is an assignment of colors to each edge so that no two adjacent edges share the same color. Various vertex and edge coloring problems have been studied in the literature [1]. A *harmonious coloring number* of a graph G is the least number of colors in a vertex coloring such that each pair of colors appears on at most one edge. Hopcroft and Krishnamoorthy [3] introduced a type of edge coloring called harmonious coloring and they also proved that the harmonious coloring problem for graphs is NP-complete. An enormous body of literature has grown around the subject Harmonious Coloring. The list of articles published on the subject can be found in [4].

In 2009, Hegde and Castelino [6] extended the concept of harmonious coloring of graphs to digraphs. The following is an extension of harmonious coloring to directed graphs.

Definition 1.1. *Let D be a directed graph with n vertices and m arcs. A function $f : V(D) \to \{1, 2, ..., k\}$, where $k \leq n$ is said to be a harmonious coloring of D if for any two arcs xy and uv of $\overrightarrow{D}$, the ordered pair $(f(x), f(y)) \neq (f(u), f(v))$. If the pair (i, i) is not assigned, then f is called a proper harmonious coloring of D. The minimum k for which D admits a proper harmonious coloring is called the proper harmonious coloring number of D and is denoted by $\overrightarrow{\chi_h}(\overrightarrow{D})$.*

In Figure 1 a proper harmonious coloring of Wagner graph and its oriented graph are given.

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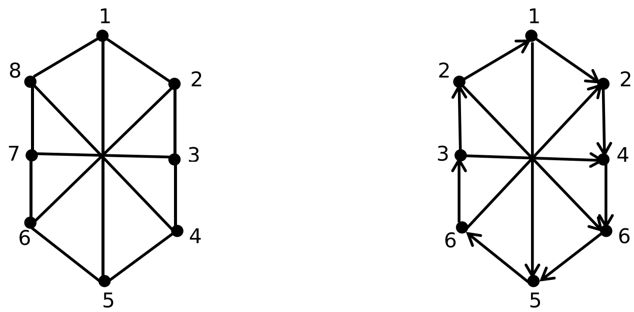

FIGURE 1. A proper harmonious coloring of Wagner Graph and its oriented graph

In general, the proper harmonious coloring problem has been viewed as an Eulerian path decomposition of graphs. Whenever such a decomposition is possible, we can find the proper harmonious coloring number of graphs [3].

Hegde and Castelino [6] gave the lower bound for the harmonious coloring number of digraphs and also found the proper harmonious coloring number of some classes of digraphs like unipath, unicycle, directed star, unicyclic wheel, complete binary tree, alternating paths and alternating cycles etc [7, 6]. They showed that the proper harmonious coloring number of any digraph with n vertices lies between $\Delta + 1$ and n. Also, they found that for any digraph $\overrightarrow{D}$, $\overrightarrow{\chi_h}(\overrightarrow{D}) \geq \lceil \frac{1+\sqrt{4m+1}}{2} \rceil$.

K. J. Edwards [5] showed that determining the exact value of the harmonious chromatic number is NP-hard for digraphs of bounded degree. Also, he has given an upper bound for the harmonious chromatic number of a general digraph. Hegde and Castelino [8] gave the lower bound for the harmonious coloring number of regular digraphs and also found the proper harmonious coloring number of some classes of regular digraphs.

The harmonious coloring number is NP-hard for digraphs. In this paper, we find the results on proper harmonious coloring of some classes of directed trees such as bistar, centipede and banana tree.

2. RESULTS

Definition 2.1. *Let $\overrightarrow{K}_{1,n}$ be the directed star with central vertex u. The directed bistar $\overrightarrow{B}_{n,n}$ is a digraph obtained by joining the central vertices say u and v of two copies of directed star $\overrightarrow{K}_{1,n}$.*

Proposition 2.2. *Let $\overrightarrow{B}_{n,n}$ be a directed bistar with $2(n+1)$ vertices. Then,*

$$\overrightarrow{\chi_h}(\overrightarrow{B}_{n,n}) = \begin{cases} \Delta + 2 & \text{when } id(u) = id(v) = od(u) = od(v) \\ \Delta + 1 & \text{otherwise} \end{cases}$$

where u and v both are central vertices and Δ is the maximum indegree or outdegree of u or v.

Proof. Let $\overrightarrow{B}_{n,n}$ be a directed bistar with two central vertices u and v having $(n+1)$ vertices in each copy of a star. Let s be the maximum indegree (id) of u or v and t be the maximum outdegree (od) of u or v. Label any central vertex (u or v) with maximum degree (id or od) as 1.

Case (a): Let $s > t$. The incoming arcs to the central vertex which has maximum indegree(u or v) will be $(2,1),(3,1),...,(s+1,1)$ and the outgoing arcs from the central vertex(u or v) will be $(1,2),(1,3),...,(1,t)$. The remaining arcs can be assigned with the other pair of colors such a way that no pair of color is repeated. Thus, $\overrightarrow{\chi_h}(\overrightarrow{B}_{n,n}) = s+1$.
Case (b): Let $t > s$. The outgoing arcs from the central vertex which has maximum outdegree(u or v) will be $(1,2),(1,3),...,(1,t+1)$ and the incoming arcs to the central vertex(u or v) will be $(2,1),(3,1),...,(s,1)$. The remaining arcs can be assigned with the other pair of colors such a way that no pair of color is repeated. Thus, $\overrightarrow{\chi_h}(\overrightarrow{B}_{n,n}) = t+1$.
From the above two cases, we can conclude that
$\overrightarrow{\chi_h}(\overrightarrow{B}_{n,n}) = max[id \text{ (u } or \text{ v)}, od \text{ (u } or \text{ v)}] + 1 = (\Delta + 1)$.
When $id(\text{u}) = id(\text{v}) = od(\text{u}) = od(\text{v})$, $s+1$ or $t+1$ colors are not enough to color all the vertices. We need one more additional color to color the vertices so that the graph represents the proper harmonious coloring. Thus, $\overrightarrow{\chi_h}(\overrightarrow{B}_{n,n}) = \Delta + 2$. □

Figure 2 is an illustration of the above result.

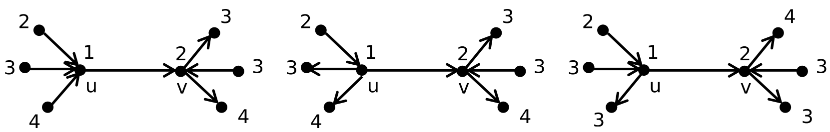

FIGURE 2. A proper harmonious coloring of directed bistar $\overrightarrow{B}_{3,3}$

Definition 2.3. *An n-centipede $\overrightarrow{CP}_n$ is a directed tree obtained by joining each vertex of the unipath to a pendent vertex whose indegree or outdegree is zero.*

Proposition 2.4. *Let $\overrightarrow{CP}_n$ be a centipede with vertices $2n$. Then,*

$$\overrightarrow{\chi_h}(\overrightarrow{CP}_n) = \begin{cases} k+1 & if\ \frac{k^2}{2} - k + 2 \leq n \leq \frac{k}{2}(k-1) \text{ and } k \text{ is even} \\ k & otherwise \end{cases}$$

where $k = \lceil \frac{1+\sqrt{8n-3}}{2} \rceil$.

Proof. Since centipede has $m = 2n-1$ arcs, $(\overrightarrow{\chi_h})(\overrightarrow{CP}_n) \geq \lceil \frac{1+\sqrt{8n-3}}{2} \rceil$. Let $k = \lceil \frac{1+\sqrt{8n-3}}{2} \rceil$ except for $\frac{k^2}{2} - k + 2 \leq n \leq \frac{k}{2}(k-1)$. Consider a complete symmetric digraph $\overleftrightarrow{K}_k$ with k vertices. Then $\overleftrightarrow{K}_k$ contains $k(k-1)$ arcs. To find the proper harmonious coloring of $\overrightarrow{CP}_n$, it is sufficient to find an Eulerian trail in $\overleftrightarrow{K}_k$. It is obvious that we get an Eulerian closed trail in each decomposition which traverses through any vertex. Hence it requires only k colors except for $\frac{k^2}{2} - k + 2 \leq n \leq \frac{k}{2}(k-1)$.
When k is even, for each color there will be an odd number of ordered pairs. But for $\frac{k^2}{2} - k + 2 \leq n \leq \frac{k}{2}(k-1)$, we require even number of ordered pairs of colors from each color for the proper harmonious coloring of vertices. Since there are only odd number of ordered pairs for each color, we require one additional color to color the vertices of $\overrightarrow{CP}_n$. Thus, $\overrightarrow{\chi_h}(\overrightarrow{CP}_n) = k+1$, when k is even and $\frac{k^2}{2} - k + 2 \leq n \leq \frac{k}{2}(k-1)$. □

Figure 3 is an illustration of the above result.

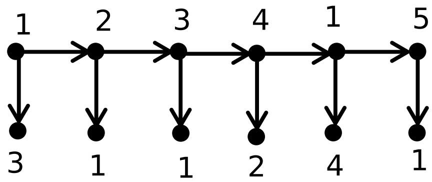

FIGURE 3. A proper harmonious coloring of centipede $\overrightarrow{CP}_6$

Definition 2.5. *Let $\overrightarrow{S}_k$ be a directed star with k vertices. The directed banana graph $\overrightarrow{BN}_{n,k}$ is a directed tree obtained by connecting one leaf of each n copies of $\overrightarrow{S}_k$ to a single vertex known as root vertex.*

Proposition 2.6. *Let $\overrightarrow{BN}_{n,k}$ (n copies of $k-star$) be a directed banana tree with $nk+1$ vertices, where u is the root vertex and $v_1, v_2, ..., v_n$ are the central vertices of $\overrightarrow{S}_k$. Then,*
(i) for $n \geq k$ and if $id(u) = 0$ or $od(u) = 0$, $\overrightarrow{\chi_h}(\overrightarrow{BN}_{n,k})$=$\Delta+1$, where Δ is the maximum indegree(id) or outdegree(od) of the root vertex.
(ii) for $n < k$ and $id(u) = 0$ or $od(u) = 0$,

$$\overrightarrow{\chi_h}(\overrightarrow{BN}_{n,k})=\begin{cases} \Delta^{'}+1 & if\ id(v_1, v_2, ..., v_n) = 0 \ or \ od(v_1, v_2, ..., v_n) = 0 \\ \Delta^{'}+2 & if\ id(v_1, v_2, ..., v_n) = 1 \ or \ od(v_1, v_2, ..., v_n) = 1 \ and \ 2n \geq k-1 \\ \Delta^{'}+1 & if\ id(v_1, v_2, ..., v_n) = 1 \ or \ od(v_1, v_2, ..., v_n) = 1 \ and \ 2n < k-1 \end{cases}$$

where $\Delta^{'}$ is the maximum indegree(id) or outdegree(od) of the central vertex.

Proof. (i) For $n \geq k$, the proof is obvious and can be easily verified.
(ii) For $n < k$, let s be the number of incoming arcs and t be the number of outgoing arcs of root vertex. Also, let $s_1, s_2, ..., s_n$ and $t_1, t_2, ..., t_n$ respectively be the number of incoming and outgoing arcs of $\overrightarrow{S}_k$.
Case (a): Let u be the root vertex and $v_1, v_2, ..., v_n$ be the central vertices of n copies of $\overrightarrow{S}_k$. Since $n < k$, the number of arcs connected to the root vertex are less than the number of arcs connected to the central vertex of any copy of $\overrightarrow{S}_k$. Hence $s \leq s_1$ or $s \leq s_2$ or ... $s \leq s_n$ and $s \leq t_1$ or $s \leq t_2$ or ...$s \leq t_n$. Let the color of the vertex v_1 be 1, then the incoming arcs of v_1 will be $(2,1), (3,1), (4,1), ..., (s_1+1, 1)$. Therefore, $\overrightarrow{\chi_h}(\overrightarrow{BN}_{n,k}) = s_1 + 1$. If v_1 has all outgoing arcs then the coloring of arcs will be $(1,2), (1,3), (1,4), ..., (1, t_1+1)$. Thus, we obtain $\overrightarrow{\chi_h}(\overrightarrow{BN}_{n,k}) = \Delta^{'} + 1$.
Case (b): Let $\overrightarrow{S}_k$ have total $k-1$ arcs (leaves), where one arc is pointing inward direction to the central vertex and remaining $k-2$ arcs are pointing outward direction (or vice versa). Take the first copy of $\overrightarrow{S}_k$. Here we need exactly $k-1$ colors to color the central vertex and the $k-2$ vertices connected to the arcs which are in the same direction (inward or outward, $\Delta^{'}(v_1) = k-2$). The remaining $n-1$ copies of $\overrightarrow{S}_k$ can be colored with the same $k-1$ colors by assigning different colors to the central vertices and the root vertex can be colored with any one of $k-1$ colors.
When $2n \geq k-1$, to color the remaining vertices which connects the root

vertex and the central vertices of $\overrightarrow{S}_k$, we cannot assign the colors from the set of $k-1$ colors which will lead to the repetition of ordered pair of colors. Hence we need one extra color to properly color the vertices. Thus, $\overrightarrow{\chi_h}(\overrightarrow{BN}_{n,k}) = \Delta' + 2$.
When $2n < k-1$, the above coloring pattern can be followed and we obtain $\overrightarrow{\chi_h}(\overrightarrow{BN}_{n,k}) = \Delta' + 1$. □

Figure 4 is an illustration of the above result.

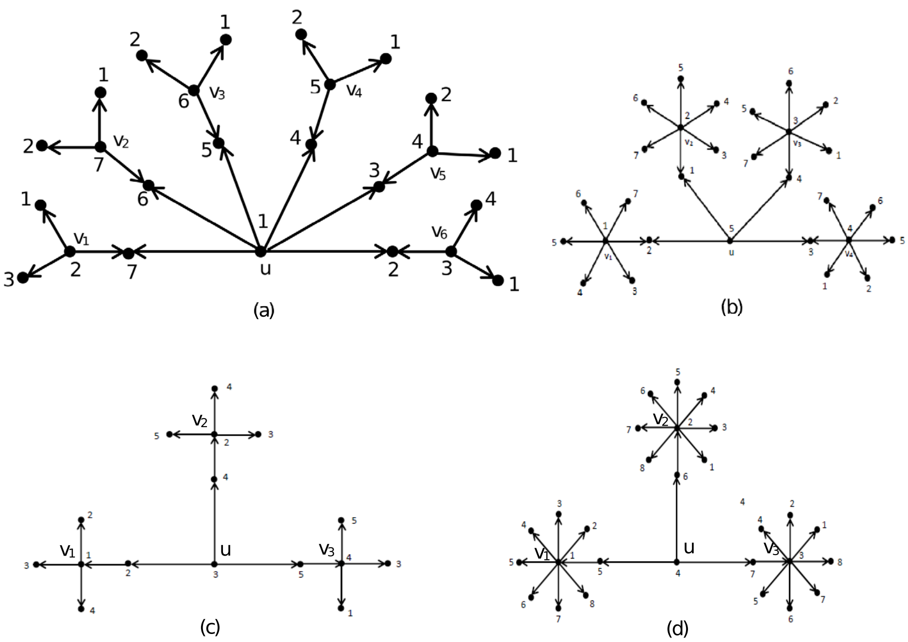

FIGURE 4. A proper harmonious coloring of different banana trees (a) $\overrightarrow{BN}_{6,4}$; (b) $\overrightarrow{BN}_{4,7}$; (c) $\overrightarrow{BN}_{3,5}$ and (d) $\overrightarrow{BN}_{3,9}$

3. Conclusion and Future Scope

The harmonious coloring has several applications. Harmonious coloring problem has potential applications in communication networks (i.e. transportation networks, computer networks etc). It has been determined that the proper harmonious coloring number is NP-hard. In this paper, the proper harmonious coloring number was found for some classes of directed trees. The proper harmonious coloring number for more classes of digraphs can be found. Also, the upper bounds may be found in terms of degree, number of vertices and edges.

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